String Topological Robotics

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Aim

Link two well known theories; namely the *string topology* (founded by M. Chas and D. Sullivan in 1999) and the *topological robotics* (founded by M. Farber some few years later, in 2003).

Context

- M : path connected manifold;
- $PM := M^{[0,1]}$: the set of all free paths in M.

Motion Planning Algorithm

Definition

A Motion Planning Algorithm (MPA for short) is any section $s : M \times M \longrightarrow PM$ of the *bi-evaluation map*

$$\begin{array}{rcccl} \mathrm{ev} & : & PM & \longrightarrow & M \times M \\ & & \gamma & \longmapsto & (\gamma(0), \gamma(1)) \end{array}$$

Why sections are algorithms

$$\text{Let } s : \begin{array}{ccc} M \times M & \longrightarrow & PM \\ (m_0, m_1) & \longmapsto & \gamma := s(m_0, m_1) \end{array} \quad \text{an MPA,}$$

since $s \circ \mathrm{ev} = \mathrm{id}$, then $\gamma(0) = m_0$ and $\gamma(1) = m_1$.

Thus

- The input is the pair $(m_0, m_1) \in M \times M$, that specifies the departure state and the arrival one;
- The output is a path $\gamma := s(m_0, m_1) \in PM$ which propose to the robot a motion in M from m_0 to m_1;
- M is viewed as all possible configurations of a mechanical system; a robot for example.

Robot Motion Stability

M. Farber, [F03]

To make the robots motions stable, meaning that close pairs (m_0, m_1) and (m'_0, m'_1) should produce close motions $s(m_0, m_1)$ and $s(m'_0, m'_1)$, M. Farber asked sections to be continuous and showed that it is equivalent to suppose M contractible.

Context

- M : oriented and closed manifold of dimension n;
- $LM := M^{S^1}$: The set of free loops in M;
- The evaluation map : $\begin{array}{rccc} \mathrm{ev} : & LM & \longrightarrow & M \\ & \gamma & \longmapsto & \gamma(0) \end{array}$.

Context

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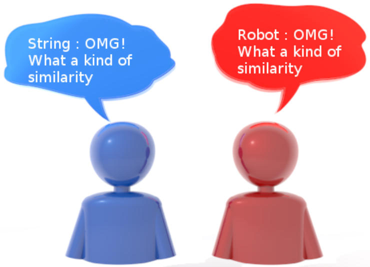


Tools

Definition

An i-chain $x : K_x \longrightarrow LM$ and a j-chain $y : K_y \longrightarrow LM$ are called transverse whenever their associated evaluation map

$$\begin{array}{rcccl} \mathrm{ev}_x \times \mathrm{ev}_y & : & K_x \times K_y & \longrightarrow & M \times M \\ & & (k_x, k_y) & \longmapsto & (x(k_x)(0), y(k_y)(0)) \end{array}$$

is transverse, in the basic context, to the diagonal map $\Delta_M : m \mapsto (m, m)$.

Tools

Usefull Remark

In this case,

$$K_{x\bullet y} := (\mathrm{ev}_x \times \mathrm{ev}_y)^{-1}(\Delta_M) \subset K_x \times K_y$$

is a manifold with corners, compact, oriented, and of dimension $i + j - n$.

Tools

Definition

The loop product $x \bullet y : K_{x \bullet y} \longrightarrow LM$ is defined by :

$$(x \bullet y)(k_x, k_y)(t) := \begin{cases} x(k_x)(2t) & 0 \leqslant t \leqslant 1/2 \\ y(k_y)(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

Tools

Usefull Remark

One may extend transversely the loop product at the level of the homology as showed by the following commutative diagram

$$\begin{array}{ccc} H_i(LM)\otimes H_j(LM) & \xrightarrow{\ \bullet\ } & H_{i+j-n}(LM) \\ \Big\downarrow{\scriptstyle \mathrm{ev}_*\otimes \mathrm{ev}_*} & & \Big\downarrow{\scriptstyle \mathrm{ev}_*} \\ H_i(M)\otimes H_j(M) & \xrightarrow{\ \cap\ } & H_{i+j-n}(M) \end{array}$$

Mains Results

M. Chas, D. Sullivan [CS99]

The down shifted graded homology

$$\mathbb{H}_*(LM) := (H_{*+n}(M), \bullet)$$

is an associative and commutative graded algebra.

Goal

Endow the homology of $\mathrm{Sect}(\mathrm{ev})$ with a string product which yield structures of Gerstenhaber algebra and Batalin-Vilkovisky algebra.

First obstruction

To insure stability of the robots motions, M. Farber asked sections to be continuous and showed that it is equivalent to suppose M contractible. In this case $\mathrm{Sect}(\mathrm{ev})$ is also contractible and its homology is trivial.

The solution

Consider a compact Lie group G acting on M.
This group action will insure the stability claimed by Farber for his motion planners without requiring sections to be continuous.

Tools

Definition

- Bi-loops :

$$LM \times_{M/G} LM := \{(\gamma, \tau) \in LM \times LM \ : \ G.\gamma(1/2) = G.\tau(0)\},$$

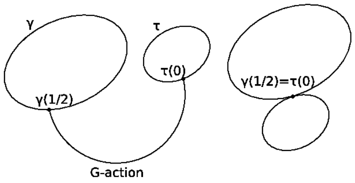
- Loop bi-evaluation map

$$\begin{array}{rccc} ev^{LP} : & LM \times_{M/G} LM & \longrightarrow & M \times M \\ & (\gamma, \tau) & \longmapsto & (\gamma(0), \tau(1/2)) \end{array}.$$

- Loop Motion Planner (LMP for short) any $G \times G$-homotopic section $s : M \times M \longrightarrow LM \times_{M/G} LM$ of ev^{LP}.

Why LMP is an algorithm

- Input : a pair of points $(m_0, m_1) \in M \times M$;
- output : a pair of loops $(\gamma, \tau) \in LM \times LM$, such that γ starts from m_0 before meeting τ in the point m_1 at time $t = 1/2$.

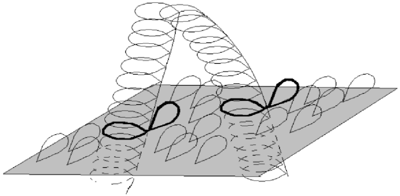


LMP Product

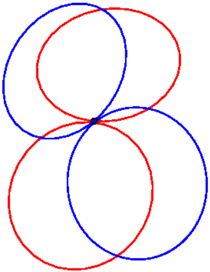
On $\mathcal{M}^{\mathrm{LP}}(M)$, the set of LMPs, one may define a natural concatenation, called *LMP product*, by putting:

$$
\begin{array}{rcll}
\mu(s_0, s_1)(m_0, m_1)(t) & = & s_0(m_0, m_1)(t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\
 & = & s_0(m_0, m_1)(3t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant \frac{2}{3} \\
 & = & s_1(m_0, m_1)(3t-2) & \text{if } \frac{2}{3} \leqslant t \leqslant 1
\end{array}
$$

What a kind of ressemblance



Chas-Sullivan product



LMP product

The LMP Homology

Definition

- *i*-simplices in $\mathcal{M}^{\mathrm{LP}}(M)$: all maps $\Sigma : \Delta^i \longrightarrow \mathcal{M}^{\mathrm{LP}}(M)$ with some additional Laudenbach requirements [Lau11];
- Boundary Operator : $\partial\Sigma := \sum_{k=0}^{i}(-1)^k F_k\Sigma$, where F_k means k-th face;
- LMP Homology : $H_*(\mathcal{M}^{\mathrm{LP}}(M); \partial)$.

Transversality

Key Tool

- *LMP bi-evaluation map*

$$\begin{array}{rccl} \mathrm{ev}_*^{\mathrm{LP}} : & \mathcal{M}^{\mathrm{LP}}(M) & \longrightarrow & M \times M \\ & s & \longmapsto & (s(-,-)(0), s(-,-)(1/2)) \end{array},$$

- Let $\Sigma : \Delta^i \longrightarrow \mathcal{M}^{\mathrm{LP}}(M)$ an i-simplex of $\mathcal{M}^{\mathrm{LP}}(M)$, and $\Theta : \Delta^j \longrightarrow \mathcal{M}^{\mathrm{LP}}(M)$ a j-one. We put $\sigma := ev_*^{\mathrm{LP}}(\Sigma) : \Delta^i \longrightarrow M^2$ and $\theta := ev_*^{\mathrm{LP}}(\Theta) : \Delta^j \longrightarrow M^2$ their associated simplicies in M^2.

Transversality

Definition

The bi-simplex $\Sigma \times \Theta$ is said to be *transverse* when $\sigma \times \theta$ and all its faces are transverse in $M^2 \times M^2$ (in the basic context) to the diagonal map

$$\Delta_{M\times M} : \begin{array}{ccc} M \times M & \longrightarrow & M^2 \times M^2 \\ (m_0, m_1) & \longmapsto & (m_0, m_1, m_0, m_1) \end{array}.$$

Transversality

Then, set $W := (\sigma \times \theta)^{-1}(\Delta_{M\times M})$ and consider the following commutative diagram

$$\begin{array}{ccc} W \subset \Delta^i \times \Delta^j & \xrightarrow{\Sigma\times\Theta} & \mathcal{M}^{\mathrm{LP}}(M) \times \mathcal{M}^{\mathrm{LP}}(M) \\ & \searrow^{\sigma\times\theta} & \downarrow{\scriptstyle \mathrm{ev}_*^{\mathrm{LP}} \times \mathrm{ev}_*^{\mathrm{LP}}} \\ & & \Delta_{M\times M} \subset M^2 \times M^2 \end{array},$$

Useful remark

W is an orientable sub-manifold of $\Delta^i \times \Delta^j$ with corners and of dimension $i + j - 2n$.

Intersection LMP product

Useful remark

One may perform the precedent commutative diagram and get the second more useful following one

$$\begin{array}{ccccc}
 & & \xrightarrow{\quad \Sigma.\Theta\ :\ \text{intersection LMP product} \quad} & & \\
W \simeq \Delta^{i+j-2n} & \xrightarrow{\Sigma\times\Theta} & \mathcal{M}^{\mathrm{LP}}(M)\times\mathcal{M}^{\mathrm{LP}}(M) & \xrightarrow{\mu} & \mathcal{M}^{\mathrm{LP}}(M) \\
 & \searrow^{\sigma\times\theta} & \downarrow{\scriptstyle \mathrm{ev}_*^{\mathrm{LP}}\times\mathrm{ev}_*^{\mathrm{LP}}} & & \\
 & & \Delta_{M\times M}\simeq M\times M & &
\end{array}$$

Intersection LMP product

Definition

- The intersection LMP product is now well defined by setting :

$$\Sigma.\Theta := \mu \circ (\Sigma \times \Theta)_{|W},$$

- It can be extended on $H_*(\mathcal{M}^{\mathrm{LP}}(X);\partial)$ to the *string LMP product* by setting

$$[\Sigma] \bullet [\Theta] := [\Sigma.\Theta].$$

Y. Derfoufi, M. [DM16]

Applying the regrading $\mathbb{H}_* := H_{*+2n}$, induces on $(\mathbb{H}_*(\mathcal{M}^{\mathrm{LP}}(X)), \bullet)$ a structure of a commutative and associative graded algebra

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M. (2018), submitted work

Goal

Endow $(\mathbb{H}_*(\mathcal{M}^{\mathrm{LP}}(X)), \bullet)$ with a structure of Gerstenhaber algebra and another one of Batalin-Vilkovisky algebra.

Denotations

We adapt our notations to that of Chas-Sullivan :

- $\Sigma : \Delta^i \longrightarrow \mathcal{M}^{\mathrm{LP}}(M)$ and $\Theta : \Delta^j \longrightarrow \mathcal{M}^{\mathrm{LP}}(M)$ will be replaced respectively by $x : K_x \longrightarrow \mathcal{M}^{\mathrm{LP}}(M)$ and $y : K_y \longrightarrow \mathcal{M}^{\mathrm{LP}}(M)$;
- $x(k_x) : \begin{array}{ccc} M \times M & \longrightarrow & LM \times_G LM \\ (m_0, m_1) & \longmapsto & x(k_x)(m_0, m_1) \end{array}$, with

$$\begin{array}{rcl} x(k_x)(m_0, m_1)(0) & = & m_0 \\ x(k_x)(m_0, m_1)(1/2) & = & m_1 \end{array}$$

Denotations

We adapt our notations to that of Chas-Sullivan :

- $\overline{x} := \mathrm{ev}_*^{\mathrm{LP}}(k_x) : K_x \longrightarrow M^2$ defined by

$$\overline{x}(k_x) := (x(k_x)(-,-)(0), x(k_x)(-,-)(1/2))$$

- $\widehat{y} : K_y \times [0,1] \longrightarrow M^2$ defined by

$$\widehat{y}(k_y, s) := (y(k_y)(-,-)(0), y(k_y)(-,-)(s)).$$

Useful remark

If $\overline{x} \times \widehat{y} : K_x \times K_y \times [0,1] \longrightarrow M^2 \times M^2$ is transverse to both Δ_{M^2} and all its faces, then the $*$ operator emerges on

$$K_{x*y} := (\overline{x} \times \widehat{y})^{-1}(\Delta_{M^2}) = K_x \times_{M^2} K_y \times [0,1]$$

by putting

$$x*y(k_x, k_y, s)(-,-)(t) := \begin{cases} y(k_y)(-,-)(2t) & 0 \leqslant t \leqslant \frac{s}{2} \\ x(k_x)(-,-)(1-2t+s) & \frac{s}{2} \leqslant t \leqslant \frac{s+1}{2} \\ y(k_y)(-,-)(2t-1) & \frac{s+1}{2} \leqslant t \leqslant 1 \end{cases}$$

LMP bracket

Definition

$$\{x, y\} := x * y - (-1)^{(|x|+1)(|y|+1)} y * x.$$

Theorem 1 : M. (2018), submitted

$\mathbb{H}_*(\mathcal{M}^{\mathrm{LP}}(M), \bullet, \{,\})$ is Gerstenhaber algebra. That means

- $\mathbb{H}_*(\mathcal{M}^{\mathrm{LP}}(M), \bullet)$ is an associative and commutative graded algebra;
- $\{,\}$ is Lie bracket of degree 1;
- $\{x, y \bullet z\} = \{x, y\} \bullet z + (-1)^{(|x|+1)|y|} y \bullet \{x, z\}$.

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Δ-operator

Definition

$$\Delta x(k_x, s)(t) := x(k_x)(t+s).$$

Theorem 2 : M. (2018), submitted

$\mathbb{H}_*(\mathcal{M}^{\mathrm{LP}}(M), \bullet, \{,\}, \Delta)$ is Batalin-Vilkovisky algebra. That means

- $\mathbb{H}_*(\mathcal{M}^{\mathrm{LP}}(M), \bullet)$ is an associative and commutative graded algebra;
- $\Delta \circ \Delta = 0$;
- $(a, b) \mapsto (-1)^{|a|}\Delta(a \bullet b) - (-1)^{|a|}\Delta a \bullet b - a \bullet \Delta b$ is a derivation of each variable.

Δ-operator

Definition

$$\Delta x(k_x, s)(t) := x(k_x)(t+s).$$

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Thank You

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